ON THE PROJECTIVE NORMALITY OF ARTIN-SCHREIER CURVES

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Abstract. In this paper we study the projective normality of certain Artin-Schreier curves Y_f defined over a field $\mathbb{F}$ of characteristic p by the equations $y^q + y = f(x)$, q being a power of p and $f \in \mathbb{F}[x]$ being a polynomial in x of degree m, with $(m, p) = 1$. Many Y_f curves are singular and so, to be precise, here we study the projective normality of appropriate projective models of their normalizations.

1. Introduction

Let $\mathbb{P}^2$ denote the projective plane over an arbitrary field $\mathbb{F}$ of characteristic p and let $q := p^s$ be a power of p ($s > 0$). Denote by $Y_f \subseteq \mathbb{P}^2$ the curve defined over $\mathbb{F}$ by the equation $y^q + y = f(x)$, where $f(x) \in \mathbb{F}[x]$ is a polynomial of degree $m > 0$. Assume $(m, p) = 1$. The function field $\mathbb{F}(x, y)$ is deeply studied in [4], Proposition 6.4.1. In particular, the function x is known to have only one pole P_∞. Denote by $\pi : C_f \to Y_f$ the normalization (which is known to be a bijection) and set $Q_\infty := \pi^{-1}(P_\infty)$. For each $s \geq 0$ the (pull-backs of the) monomials $x^i y^j$ such that

$$i \geq 0, \quad 0 \leq j \leq q - 1, \quad qi + mj \leq s$$

form a basis of the vector space $L(sQ_\infty)$ (see [4], Proposition 6.4.1 again). The genus, g, of the curve Y_f (which is by definition the genus of the normalization C_f) is known to be $g = (m - 1)(q - 1)/2$. In this paper we study the projective normality of certain embeddings (X_f) of C_f curves into suitable projective spaces. Let us briefly discuss the outline of the paper.

- Section 2 recalls a basic definition and contains a preliminary lemma.
- In Section 3 we take an integer $m \geq 2$ which divides $q - 1$ and consider the curve C_f embedded by $L(qQ_\infty)$ into the projective space $\mathbb{P}^r$, $r := (q - 1)/m + 1$. We show that this curve is in any case projectively normal and we compute the dimension of the space of quadric hypersurfaces containing it.
- In Section 4 we pick out an integer $m \geq 2$ which divides $tq + 1$ (t being any positive integer) and consider the curve C_f embedded by $L((tq + 1)Q_\infty)$. We show that if $(tq - 1)/m \leq q - 1$ and $f(x) = x^m$ then the cited curve is in any case projectively normal.

Notice that the curve C_f and the line bundles $\mathcal{L}(qQ_\infty)$, $\mathcal{L}((tq+1)Q_\infty)$ are defined over any field $\mathbb{F} \supseteq \mathbb{F}_p$ containing the coefficients of the polynomial $f(x)$. Hence when $f(x) = x^m$ any field of characteristic p may be used.

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2. Preliminaries

In this section we recall a basic definition and prove a general lemma. The result provides in fact sufficient conditions for the projective normality of a C_f curve as defined in the Introduction.

Definition 1. A smooth curve $X \subseteq \mathbb{P}^r$ defined over a field $\mathbb{F}$ is said to be **projectively normal** if for any integer $d \geq 2$ the restriction map

$$\rho_{d,X} : S^d(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))) \to H^0(X, \mathcal{O}_X(d))$$

is surjective, S^d denoting the symmetric d-power of the tensor product.

Lemma 2. Consider a C_f curve as in Section 1 (q, m and f being as in the definitions). Set $C := C_f$. Fix integers a, b, e such that $a \geq 0$, $b \geq 0$, $a+b > 0$ and $e \geq aq+bm+(m-1)(q-1)-1$. The multiplication map

$$\mu : L((aq+bm)Q_\infty) \otimes L(eQ_\infty) \to L((e+aq+bm)Q_\infty)$$

is surjective.

Proof. As we will explain below, this is just a particular case of the base-point free pencil trick ([1], p. 126). Since the Weierstrass semigroup of non-gaps of Q_∞ contains m and q, the line bundle $\mathcal{O}_C((aq+bm)Q_\infty)$ is spanned by its global sections. Since $(aq+bm) > 0$, we have $h^0(C, \mathcal{O}_C((aq+bm)Q_\infty)) \geq 2$. Hence there is a two-dimensional linear subspace $V \subseteq H^0(C, \mathcal{O}_C((aq+bm)Q_\infty))$ (defined over $\overline{\mathbb{F}}$) spanning $\mathcal{O}_C((aq+bm)Q_\infty)$. Taking a basis, say $\{w_1, w_2\}$, of V we get an exact sequence of line bundles on C (over $\overline{\mathbb{F}}$):

$$0 \to \mathcal{O}_C((e-aq-bm)Q_\infty) \to \mathcal{O}_C(eQ_\infty)^{\oplus 2} \xrightarrow{\phi} \mathcal{O}_C((e+aq+bm)Q_\infty) \to 0$$

in which ϕ is induced by the multiplication by the column vector (w_1, w_2). By assumption $e - aq - bm > 2g-2$. Hence $h^1(C, \mathcal{O}_C((e-aq-bm)Q_\infty)) = 0$. It follows that the map

$$\psi : H^0(C, \mathcal{O}_C(eQ_\infty))^{\oplus 2} \to H^0(C, \mathcal{O}_C((e+aq+bm)Q_\infty))$$

induced in cohomology by the map ϕ of the previous exact sequence is surjective. Since $V \subseteq H^0(C, \mathcal{O}_C((aq+bm)Q_\infty))$, μ is surjective. $\square$

In the following sections the previous result will be applied to appropriate embeddings of C_f curves.

3. The case $m|q-1$

Assume that $m \geq 2$ is an integer which divides $q-1$ and set $c := (q-1)/m$. If $c = 1$ then Y_f is a smooth plane curve and it is of course projectively normal. Hence we can focus on the case $c \geq 2$. Notice that the point $P_\infty \in Y_f(\mathbb{F})$ defined in the Introduction is the only singular point of Y_f, for any choice of $f(x)$ as in the definitions. We have also an identity of vector spaces $H^0(C_f, \pi^*(\mathcal{O}_{Y_f}(1))) = L(qQ_\infty)$ and by the results stated in the Introduction it can be easily seen that a basis of them is $\{1, x, y, ..., y^c\}$ ($c \leq q-1$ here). Since the linear system spanned by $\{1, x, y\}$ is base-point free (it is also the linear system inducing the composition of π with the inclusion of Y_f in the plane) then also the complete linear system $L(qQ_\infty)$ is base-point free. Hence it defines a morphism $\varphi : C_f \to \mathbb{P}^r$, $r = c+1$.

Remark 3. Since π is injective and has invertible differential at any point of $C_f \setminus \{Q_\infty\}$, then also φ is injective with non-zero differential at any point of $C_f \setminus \{Q_\infty\}$. Finally, the

differential of φ is non-zero even in Q_∞. Indeed, since $L(qQ_\infty)$ has no base-points, in order to prove that φ has a non-zero differential at Q_∞ it is sufficient to prove that

$$h^0(C_f, (q-2)Q_\infty) = h^0(C_f, qQ_\infty) - 2.$$

To do this, we may notice that a basis of $L(qQ_\infty)$ is given by a basis of $L((q-2)Q_\infty)$ and the monomial x and y^c (see the Introduction).

By the previous remark we get that φ is in fact an embedding of C_f into $\mathbb{P}^r$. Set $X_f := \varphi(C_f)$ and, for any integer $s \geq 1$, denote by

$$\mu_s : L(qQ_\infty) \otimes L(sqQ_\infty) \to L(((s+1)q)Q_\infty)$$

the multiplication map.

Lemma 4. If μ_s is surjective for all $s \geq 1$ then X_f is projectively normal.

Proof. Fix an integer $t \geq 2$ and assume that μ_s is surjective for all $s \in \{1, \ldots, t-1\}$. We need to prove the surjectivity of the linear map $\rho_t : S^t(L(qQ_\infty)) \to L(tqQ_\infty)$. Notice that in arbitrary characteristic $S^t(L(qQ_\infty))$ is defined as a suitable quotient of $L(qQ_\infty)^{\otimes t}$ ([**?**], §A2.3), i.e. $\tau_t = \rho_t \circ \eta_t$, where $\tau_t : L(qQ_\infty)^{\otimes t} \to L(tqQ_\infty)$ is the tensor power map and $\eta_t : L(qQ_\infty)^{\otimes t} \to S^t(L(qQ_\infty))$ is a surjection. Hence ρ_t is surjective if and only if τ_t is surjective. Since $\tau_2 = \mu_1$, τ_2 is surjective. So assume $t > 2$ and that τ_{t-1} is surjective. Since τ_{t-1} and μ_{t-2} are surjective, τ_t is surjective. □

Proposition 5. If $s \geq m$ then μ_s is surjective.

Proof. If $s \geq m$ then $sq \geq q + (m-1)(q-1) - 1$. Apply Lemma 2 by setting $e := sq$, $a := 1$ and $b := 0$. □

Theorem 6. The curve X_f is projectively normal.

Proof. By Lemma 8 it is enough to prove that μ_s is surjective for all $s \geq 1$. The case $s \geq m$ is covered by Proposition 5. So let us assume $1 \leq s < m$. Let i, j be integers such that $i \geq 0$, $0 \leq j \leq q-1$ and $qi + mj \leq (s+1)q$.

- If $qi + mj \leq sq$ then $x^i y^j$ is in the image of μ_s because $1 \in L(qQ_\infty)$.
- If $sq < qi + mj \leq (s+1)q$ and $i > 0$ then $x^{i-1}y^j \in L(sqQ_\infty)$. Since $x \in L(qQ_\infty)$ then $x^i y^j$ is in the image of μ_s.
- If $i = 0$ and $sq < mj < (s+1)q$ then $j > sq/m = s(c + 1/m) > c$ and $mj \leq (s+1)q - 1$. By the latter inequality we get $m(j-c) \leq (s+1)q - 1 - mc$. Observe that $(s+1)q - 1 - mc = sq$ and so $m(j-c) \leq sq$. This proves that $y^{j-c} \in L(sqQ_\infty)$. Finally, $\mu_s(y^c \otimes y^{j-c}) = y^j$.
- If $i = 0$ and $mj = (s+1)q$ then $j = (s+1)q/m = (s+1)(c+1/m) = (s+1)c + (s+1)/m$. Since $0 \leq j \leq q-1$ is a nonnegative integer, we must have $(s+1)/m \in \mathbb{N}$. Since $1 \leq s < m$ we get $s = m-1$. It follows $mj = mq$ and $j = q$, a contradiction.

This proves the theorem. □

Corollary 7. Assume $m \geq 3$. The curve $X_f \subseteq \mathbb{P}^{c+1}$ is contained into $\binom{c+3}{2} - 3c - 3$ linearly independent quadric hypersurfaces.

Proof. In the notations of Definition 1 set $X := X_f$ and $d := 2$. Define $r := c+1$. By Theorem 6 the restriction map

$$\rho_{2,X_f} : S^2(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))) \to H^0(X_f, \mathcal{O}_{X_f}(2))$$

is surjective. Hence, in particular, the restriction map

$$\rho : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(X_f, \mathcal{O}_{X_f}(2))$$

is surjective. Since $m \geq 3$ (by assumption) we can easily check that a basis of the vector space $L(2qQ_\infty)$ consists of the following monomials:

$$\{1, y, ..., y^{2c}, x, xy, ..., xy^c, x^2\}.$$

Hence $h^0(X_f, \mathcal{O}_{X_f}(2)) = \dim_{\mathbb{F}} \ L(2(q-1)Q_\infty) = 3c+3$. The kernel of ρ is exactly the space of the quadrics in $\mathbb{P}^r$ vanishing on X_f. By the surjectivity of ρ we easily deduce its dimension:

$$\dim_{\mathbb{F}} H^0(\mathbb{P}^r, \mathcal{I}_{X_f}(d)) = \binom{r+2}{2} - (3c+3) = \binom{c+3}{2} - 3c - 3.$$

The result follows. □

4. The case $m|tq+1$

Pick out any integer $t \geq 1$ and assume that $m \geq 2$ is a divisor of $tq+1$. Set $c := (tq+1)/m$. As in Remark 3, it can be checked that $L((tq+1)Q_\infty)$ defines an embedding, say φ, of C_f into $\mathbb{P}^r$, $r := \dim L((tq+1)Q_\infty) - 1$. Define $X_f := \varphi(C_f)$. For any integer $s \geq 1$ denote by

$$\mu_s : L((tq+1)Q_\infty) \otimes L(s(tq+1)Q_\infty) \to L((s+1)(tq+1)Q_\infty)$$

the multiplication map. As in Section 3, the projective normality of X_f is controlled by the μ_s maps.

Lemma 8. If μ_s is surjective for all $s \geq 1$ then X_f is projectively normal.

Proof. Take the proof of Lemma 4. □

Proposition 9. If $s \geq m$ then μ_s is surjective.

Proof. Apply Lemma 2 by setting $a := 0$, $b := c$ and $e := s(tq+1)$. □

In the following part of the section we focus on the case $f(x) = x^m$. In particular we are going to show that X_f curves obtained with this choice of f are projectively normal for any choice of $t \geq 1$, provided that $c \leq q-1$.

Remark 10. The assumption $c \leq q-1$ is not so restrictive from a geometric point of view. In fact, for any fixed q, the genus of X_f is $g = (q-1)(m-1)/2$. Even if c is small, here we study many curves of interesting genus.

Lemma 11. Set $f(x) := x^m$ and assume $c \leq q-1$. Pick out an integer $b \geq 0$ such that $b \leq (s+1)c$. Then y^b is in the image of μ_s.

Proof. Since $c \leq q-1$ we get $y^c \in L((tq+1)Q_\infty)$. In particular, if $b \leq c$ then we are done. Assume $b > c$. Let us prove the lemma by induction on s. If $s = 1$ then $b \leq 2c$ and $b - c \leq c \leq q-1$. Hence $y^{b-c} \in L((tq+1)Q_\infty)$ and so $y^b = \mu_1(y^c \otimes y^{b-c})$ is of course in the image of μ_1. If $s > 1$ then write $b = hc + r$ with $h \leq s$ and $0 \leq r \leq c$. Since $b - r = hc \leq sc$ we have that y^{b-r} is in the image of μ_{s-1}. In particular, it is in $L(s(tq+1)Q_\infty)$. Since $y^r \in L((tq+1)Q_\infty)$ we get $y^b = \mu_s(y^r \otimes y^{b-r})$. It follows that y^b is in the image of μ_s. □

Theorem 12. Set $f(x) = x^m$ and assume $c \leq q-1$. Then X_f is projectively normal.

Proof. By Lemma 8 it is enough to show that μ_s is surjective for any $s \geq 1$. By Proposition 9 we have only to prove that μ_s is surjective for any $1 \leq s < m$. Let i, j be integers such that $i \geq 0$, $0 \leq j \leq q-1$ and $qi + mj \leq (s+1)(tq+1)$. We will examine separately the case $2 \leq s < m$ and the case $s = 1$.

To begin with, assume $2 \leq s < m$.

- If $j \geq c$ then $x^i y^{j-c} \in L(s(tq+1)Q_\infty)$. Since $y^c \in L((tq+1)Q_\infty)$ we have $x^i y^j = \mu_s(y^c \otimes x^i y^{j-c})$.
- If $0 \leq j < c$ and $qi + mj \leq s(tq+1)$ then $x^i y^j$ is in the image of μ_s because $1 \in L((tq+1)Q_\infty)$.
- Assume $0 \leq j < c$ and $s(tq+1) < qi + mj \leq (s+1)(tq+1)$. We have $i \geq t$. Indeed, assume by absurd that $i < t$. Then

$$\begin{aligned} qi + mj &< tq + mj \\ &< tq + mc \\ &= tq + tq + 1 \\ &\leq stq + 1 \\ &< s(tq+1), \end{aligned}$$

  a contradiction (here we used $s \geq 2$).
  (A) If $qi+mj < (s+1)(tq+1)$ then $x^{i-t}y^j \in L(s(tq+1)Q_\infty)$ and $x^i y^j = \mu_s(x^t \otimes x^{i-t} y^j)$.
  (B) Assume $qi + mj = (s+1)(tq+1)$. Since $(m, p) = 1$ we have $i = am$ for an integer $a > 0$ and $j = (s+1)c - aq$. Observe that $x^i y^j = x^{am} y^{(s+1)c-aq} = (y^q + y)^a y^{(s+1)c-aq}$, which is a sum of monomials of the form y^b with $b \leq (s+1)c$. Apply Lemma 11 and the fact that μ_s is linear to get that $x^i y^j$ is in its image.

Now assume $s = 1$.

- Assume $j \geq c$. Since $qi+mj \leq 2(tq+1)$ we get $qi+m(j-c) \leq 2(tq+1)-(tq+1) = tq+1$. Hence $x^i y^{j-c} \in L((tq+1)Q_\infty)$. Finally, $\mu_1(y^c \otimes x^i y^{j-c}) = x^i y^j$.
- Assume $j < c$ and $i \geq t$. Since $qi+mj \leq 2(tq+1)$ we get $q(i-t)+mj \leq 2(tq+1)-tq = tq+2$.
  (C) If $q(i-t) + mj \leq tq + 1$ then $x^i y^j = \mu_1(x^t \otimes x^{i-t} y^j)$.
  (D) Assume $q(i-t) + mj = tq + 2$, i.e. $qi + mj = 2tq + 2$. Repeat the proof of case (B) with $s := 1$.
- Assume $j < c$ and $i < t$. Then $x^i, y^j \in L((tq+1)Q_\infty)$ and we easily get $x^i y^j = \mu_1(x^i \otimes y^j)$.

The proof is concluded. □

Remark 13. If $t = 1$ then the assumption $c \leq q-1$ is trivially satisfied (we assumed $m \neq q+1$). In this case the curve $y^q + y = x^m$ is covered by the Hermitian curve.

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